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Published in:
Lagrangian and Hamiltonian Methods for Nonlinear Control 2006

Towards Power-Based Control Strategies for a Class of Nonlinear Mechanical Systems

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Summary. In the present work we are interested on the derivation of power-based passivity properties for a certain class of non-linear mechanical systems. While for general mechanical systems, it is of common use to adopt a storage function related to the *system's energy* in order to show passivity and stabilize the system on a desired equilibrium point(e.g., IDA-PBC [1]), we want here to obtain similar properties related to the *system's power*. The motivation arises from the idea that in some engineering applications(satellite orbit motion, aircraft dynamic,etc...)seems more sensible to cope with the power flowing into the system instead of the energy that, for stabilization purposes, means to consider the systems's equilibrium the state for which the energy flow-rate(i.e.,system's power)achieve its minimum. In this respect, we recall first the power-based description for a certain class of (non)-linear mechanical systems given in [2] and then we give sufficient conditions to obtain power-based passivity properties, provided a suitable choice of port-variables. We conclude with the example of the inverted pendulum on the cart.

1 Introduction

In a previous work of the authors [2] an electrical interpretation of the motion equations of mechanical systems moving in a plane has been provided via the Brayton-Moser equations. In particular, it is proved that under certain generic assumptions the system's behavior derived from its Lagrangian function can be alternatively described through a power-based representation in an electrical fashion. It can be viewed as an extension of the well-known analogy mass/inductor, spring/capacitor and damper/resistor for linear mechanical systems to a larger class of (possibly) nonlinear systems. *The double pendulum* and *the inverted pendulum on the cart* are the illustrative examples which have been studied and electrically interpreted as nonlinear RLC circuits.

We are here interested on exploiting this power-based description for such mechanical system class in order to achieve a new passivity property using as port variables the external forces/torques and the linear/angular acceleration,and with the storage function being related to the system's power.

F. Bullo et al. (Eds.): Lag. & Hamil. Methods for Nonlin. Ctrl., LNCIS 366, pp. 123–133, 2007.
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In Section 2 we will first recall the fundamentals of Euler-Lagrange(EL) and Brayton-Moser(BM) equations in the standard form. Via the introduction of the pseudo-inductor the Brayton-Moser equations can be extended to a large class of non-linear mechanical systems, [2]. This is reviewed in Section 3, and followed by the presentation of the main result. Taking inspiration from [3] we provide a method to generate storage function candidates based on the power. We give sufficient conditions to show the power-based passivity properties. We conclude the paper in section 4 with the example of the *the inverted pendulum on the cart* for which our passivity conditions have a clear physical meaning.

2 Preliminaries

2.1 Euler-Lagrange Systems (EL)

The standard Euler-Lagrange equations (e.g., [1]) for an r degrees of freedom mechanical system with generalized coordinates $q \in \mathbb{R}^r$ and external forces $\tau \in \mathbb{R}^r$ are given by

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}(q,\dot{q})}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}(q,\dot{q})}{\partial q} = \tau \tag{1}$$

where

$$\mathcal{L}(q,\dot{q}) \triangleq \mathcal{T}(q,\dot{q}) - \mathcal{V}(q) \tag{2}$$

is the so-called Lagrangian function, $\mathcal{T}(q,\dot{q})$ is the kinetic energy which is of the form

$$\mathcal{T}(q,\dot{q}) = \frac{1}{2}\dot{q}^T D(q)\dot{q}, \tag{3}$$

where $D(q) \in \mathbb{R}^{r\times r}$ is a symmetric positive definite matrix, and $\mathcal{V}(q)$ is the potential function which is assumed to be bounded from below. Furthermore, dissipative elements can be included via the Rayleigh dissipation function as part of the external forces.

2.2 RLC-Circuits: The Brayton-Moser Equations (BM)

The electrical circuits considered in this paper are complete RLC-circuits in which all the elements can be nonlinear. The standard definitions of respectively inductance and capacitance matrices are given by

$$L(i_\rho) = \frac{\partial \phi_\rho(i_\rho)}{\partial i_\rho}\ ,\ C(v_\sigma) = \frac{\partial q_\sigma(v_\sigma)}{\partial v_\sigma}$$

where $i_\rho \in \mathbb{R}^r$ represents the currents flowing through the inductors and $\phi_\rho(i_\rho) \in \mathbb{R}^r$ is the related magnetic flux vector. On the other hand $v_\sigma \in \mathbb{R}^s$ defines the voltages across the capacitors and the vector $q_\sigma(v_\sigma) \in \mathbb{R}^s$ represents the charges stored in the capacitors. From [4] we know that the differential equations of such electrical circuits have the special form

$$Q(x)\dot{x} = \nabla P(x) \tag{4}$$

where $x = (i_\rho, v_\sigma) \in \mathbb{R}^{r+s}$, $\nabla = (\partial/\partial i_\rho, \partial/\partial v_\sigma)^T$, and

$$Q(x) = \begin{bmatrix} -L(i_\rho) & 0 \\ 0 & C(v_\sigma) \end{bmatrix}. \tag{5}$$

Furthermore the mixed potential function $P(x)$ which contains the interconnection and resistive structure of the circuit is defined as

$$P(x) = -F(i_\rho) + G(v_\sigma) + {i_\rho}^T \Lambda v_\sigma. \tag{6}$$

$F : \mathbb{R}^r \to \mathbb{R}$ and $G : \mathbb{R}^s \to \mathbb{R}$ being the current potential (content) related with the current-controlled resistors (R) and the voltage potential (co-content) related with the voltage-controlled resistors (i.e., conductors, G), respectively. More specifically, the content and co-content are defined by the integrals

$$\int_0^{i_\rho} \hat{v}_R(i'_\rho) di'_\rho, \quad \int_0^{v_\sigma} \hat{i}_G(v'_\sigma) dv'_\sigma,$$

where $\hat{v}_R(i_\rho)$ and $\hat{i}_G(v_\sigma)$ are the characteristic functions of the (current-controlled) resistors and conductors (voltage-controlled resistors), respectively. The $r \times s$ matrix Λ is given by the interconnection of the inductors and capacitors, and the elements of Λ are in $\{-1, 0, 1\}$.

2.3 Definitions

In order to introduce the electrical counter part of the position dependent mass we introduce the so-called *pseudo-inductor*. This is an inductor, but now relating the magnetic flux linkages to current and the voltage, which differs from the "usual" electrical case, i.e.,

$$\phi = f_\phi(x). \tag{7}$$

where $\phi \in \mathbb{R}^r$ is the flux related to the inductors. This definition lead to the following implicit relation between voltage and current

$$v_\rho = \frac{d\phi}{dt} = \frac{\partial f_\phi}{\partial i_\rho}\frac{di_\rho}{dt} + \frac{\partial f_\phi}{\partial v_\sigma}\frac{dv_\sigma}{dt}. \tag{8}$$

Now, define the *pseudo-inductance matrix* and the *co-pseudo-inductance matrix* as

$$\widetilde{L}(x) = \frac{\partial f_\phi}{\partial i_\rho} \quad , \quad \widetilde{M}(x) = \frac{\partial f_\phi}{\partial v_\sigma}$$

respectively, then (8) can be written as

$$v_\rho = \widetilde{L}(x)\frac{di_\rho}{dt} + \widetilde{M}(x)\frac{dv_\sigma}{dt}. \tag{9}$$

Similarly, we will consider a *capacitor* as a function relating the charge and the voltage, i.e.,

$$q_{\sigma j} = f_j^v(v_{\sigma j}) \ , \ j = 1, \ldots, s. \tag{10}$$

By defining the non-negative capacitance matrix

$$C(v_\sigma) = diag\left[\frac{\partial f_j^v(v_{\sigma j})}{\partial v_{\sigma j}}\right] \ , \ j = 1, \ldots, s,$$

we have from differentiation of (10) that

$$i_\sigma = C(v_\sigma)\frac{dv_\sigma}{dt}. \tag{11}$$

3 Power-Based Description for a Class of Mechanical Systems

In [2] the authors enlarged the class of mechanical systems for which an electrical interpretation can be provided replacing the generalized coordinates vector $(\dot{q}, q) \in \mathbb{R}^{2r}$ by the electrical states vector $(i_\rho, v_\sigma) \in \mathbb{R}^{r+s}$. In order to make the following relation a one-to-one mapping the equivalent circuit has to present a number of inductors r equal to the capacitors s. Moreover, all conservative forces acting on the masses should be (locally) invertible functions of its angular or linear position. The main result of [2] is as follows.

Theorem 1. *Consider the general Lagrangian function* (2). *Assume that:*

A1. (interconnection) $i_\rho = i_\sigma$, [1]
A2. (force-position link) $q_{\sigma j} = f_j^v(v_{\sigma j}) \in \mathcal{C}^1$ *with* $j = 1, \ldots, r$ *is a set of invertible functions such that:*

- $\frac{\partial f_j^v(v_{\sigma j})}{\partial v_{\sigma j}} = C_j(v_{\sigma j})$,
- $f_j^q(q_{\sigma j}) = v_{\sigma j}$.

Then: the Euler-Lagrange (1) *equations can be rewritten in terms of the Brayton-Moser framework as follows*

$$\begin{bmatrix} -\widetilde{D}(v_\sigma) & -[\bar{D}(x) - \hat{D}(x)]C(v_\sigma) \\ 0 & C(v_\sigma) \end{bmatrix} \begin{bmatrix} \frac{di_\rho}{dt} \\ \frac{dv_\sigma}{dt} \end{bmatrix} = \begin{bmatrix} \frac{\partial P(x)}{\partial i_\rho} \\ \frac{\partial P(x)}{\partial v_\sigma} \end{bmatrix}$$

[1] Implying that $s = r$ and $\Lambda = I$. See Remark 4 of [2] for the physical implications.

with

$$P(x) = -F(i_\rho) + G(v_\sigma) + {i_\rho}^T v_\sigma,$$

being the mixed potential function $P(i_\rho, v_\sigma)$ *and where*

$$\hat{D}(i_\rho, v_\sigma) = \begin{bmatrix} \frac{1}{2}{i_\rho}^T \frac{\partial D(q_\rho)}{\partial q_{\rho_1}}|_{q_\rho = f^v(v_\sigma)} \\ \dots \\ \frac{1}{2}{i_\rho}^T \frac{\partial D(q_\rho)}{\partial q_{\rho_r}}|_{q_\rho = f^v(v_\sigma)} \end{bmatrix} \tag{12}$$

$$C(v_\sigma) = diag\left[\frac{\partial f_j^v(v_{\sigma j})}{\partial v_{\sigma j}}, j = 1, \dots, r\right] \tag{13}$$

$$\widetilde{D}(v_\sigma) = D(q_\rho)|_{q_\rho = f^v(v_\sigma)} \tag{14}$$

$$\bar{D}(i_\rho, v_\sigma) = \begin{bmatrix} a_{11}(i_\rho, v_\sigma) & \cdots & a_{1r}(i_\rho, v_\sigma) \\ \vdots & \ddots & \vdots \\ a_{r1}(i_\rho, v_\sigma) & \cdots & a_{rr}(i_\rho, v_\sigma) \end{bmatrix} \tag{15}$$

with $a_{ij}(i_\rho, v_\sigma) = {i_\rho}^T C^{-1}(v_\sigma) \nabla_{v_\sigma} \widetilde{D}_{ij}(v_\sigma)$ *for* $\quad i, j \in \{1, r\}$.

Corollary 1. *As a consequence of Theorem 1, recalling the definitions of the pseudo-inductor and the capacitor adopted in* (9) *and* (11) *respectively, the BM equations can be then re-written in the following more compact form*

$$\widetilde{Q}(x)\dot{x} = \nabla P(x) \tag{16}$$

with

$$\widetilde{Q}(x) = \begin{bmatrix} -\widetilde{L}(v_\sigma) & -\widetilde{M}(i_\rho, v_\sigma) \\ 0 & C(v_\sigma) \end{bmatrix}$$

and where $\widetilde{L}(v_\sigma) = \widetilde{D}(v_\sigma)$, $\widetilde{M}(i_\rho, v_\sigma) = \left[\dot{\widetilde{D}}(v_\sigma) - \hat{D}(i_\rho, v_\sigma)\right] C(v_\sigma)$.

Remark 1. The former result can been interpreted in two ways. From one side we established under which conditions–A1 and A2–a mechanical systems described by EL equations, through derivation of an *energy-based* function called Lagrangian, has a clear electrical counterpart based on the classical states analogy force/voltage and speed/current. On the other side, we state that this class of mechanical systems that can be electrically interpretable yields a *power-based* description in the BM framework. Under this second perspective we will present, in the further section, our main result.

3.1 Power-Based Passivity Properties

This section is dedicated to the derivation of passivity *sufficient* conditions for that class of mechanical systems that admits the power-based description given in (16). For that we have to find a *storage function candidate* and a corresponding set of *port variables*. It is then instrumental for the derivation of the next theorem to re-define the mixed-potential function $P(x)$ extracting the voltage sources $v_s \in \mathbb{R}^l$ with $l \leq r$, from the content term $F(i_\rho)$ as follows

$$P(x) = \widetilde{P}(x) - x^T B v_s \tag{17}$$

with $B = (B_s, 0)^T$ and $B_s \in \mathbb{R}^{r \times l}$.

Remark 2. In equation (4) we restricted our analysis to circuits having only voltage sources in series with inductors. This choice seems to be sensible considering that the mechanical counterpart of a current source is a velocity source which have no clear sense from a physical view point.

Storage Function Candidate

Following the procedure of [3], we can pre-multiply (16) by $\dot{x}^T$ obtaining

$$\dot{x}^T \widetilde{Q}(x) \dot{x} = \dot{x}^T \nabla_x \widetilde{P}(x) - \dot{x}^T B v_s$$

that can be re-arranged as follows

$$\frac{d\widetilde{P}}{dt}(x) = \dot{x}^T B v_s + \dot{x}^T \widetilde{Q}(x) \dot{x} \tag{18}$$

and which consists of the sum of two terms. The first one represents the inner product of the source variables in the suited form $\dot{x}^T B v_s = v_s^T \dot{\hat{i}}_s$, where we assume the vector $i_s \in \mathbb{R}^l$ indicating the correspondent current terms flowing from each inductor series-connected voltage source.

The second one is a quadratic term. In general $\widetilde{Q}(x)$ is not symmetric and its symmetric part is sign indefinite making difficult the derivation of the power-balance inequality we are looking for. In order to overcome this drawback we follow the same procedure exploited in [3],[4],[5] that basically provides a method to describe the system (16) by another admissible pair, say $\widetilde{Q}_a(x)$ and $P_a(x)$. For instance, if the new pair fulfills the following conditions:

C1. $\widetilde{Q}_a^T(x) + \widetilde{Q}_a(x) \leq 0$

C2. $\widetilde{P}_a(x) : \mathbb{R}^{s+r} \to \mathbb{R}$ is positive semi-definite scalar function

we may state that

$$\frac{d\widetilde{P}_a}{dt}(x) \leq \dot{x}^T B v_s \tag{19}$$

being $\widetilde{P}_a(x)$ the storage function candidate related to $P_a(x)$ by (17), the pair $(v_s, \dot{\hat{i}}_s)$ is passive and can serve as port-variables.

Power-Balance Inequality and Passivity Requirements

In the next theorem we will provide some conditions for passivity that may be useful for control in the power-based framework. In particular, we refer to a previous work of the second author [3] where the storage function has the dimension of power and is defined as a re-shaped mixed potential function $\widetilde{P}_a(x)$. This new function is then related to a new matrix $\widetilde{Q}_a(x)$ and both, having common solutions for (16), are related to the original pair $\widetilde{Q}(x), \widetilde{P}(x)$ by the following relations[2]

$$\begin{aligned} \widetilde{Q}_a(x) &= \left\{\lambda I + \tfrac{1}{2}\nabla^2 \widetilde{P}(x)\Pi(x) + \tfrac{1}{2}\nabla[\nabla^T \widetilde{P}(x)\Pi(x)]\right\} \widetilde{Q}(x) \\ \widetilde{P}_a(x) &= \lambda\widetilde{P}(x) + \tfrac{1}{2}\nabla^T \widetilde{P}(x)\Pi(x)\nabla\widetilde{P}(x) \end{aligned}$$

with $\Pi(x) \in \mathbb{R}^{r\times r}$ a symmetric matrix and $\lambda \in \mathbb{R}$ any constant.

Theorem 2. *Consider an electrical system for which the dynamics is described by* (16) *and assume A1 and A2 hold. Moreover, Assume that*

A3. (positivity) *pseudo-inductors and capacitors matrices are positive definite*
A4. (linearity in the content) $F(i_\rho) = -(1/2){i_\rho}^T R_\rho i_\rho$ *with the current-controlled resistor matrix* R_ρ *being constant and positive definite*
A5. (damping condition)

$$\left\|-2R_\rho^{-1}\widetilde{M}(x)C^{-1}(v_\sigma) + \widetilde{M}^T(x)\widetilde{L}^{-1}(v_\sigma)\widetilde{M}(x)C^{-1}(v_\sigma) + \beta(x)\right\| \leq 1$$

with

$$\beta(x) = \frac{\partial}{\partial v_\sigma}\left[{i_\rho}^T \widetilde{L}(v_\sigma)R_\rho^{-1}C^{-1}(v_\sigma)\right].$$

A6. (technical assumption)

$$\widetilde{L}(v_\sigma)R_\rho^{-1}C^{-1}(v_\sigma) \geq 0$$

then

$$\int_0^t v_s^T(t')\frac{di_s}{dt'}dt' \geq \widetilde{P}_a(x(t)) - \widetilde{P}_a(x(0)). \tag{20}$$

Proof. First, we set the matrix $\Pi(x)$ and the scalar λ in order to guarantee the semi-definite positivity of the storage function $\widetilde{P}_a(x)$ and to satisfy the following requirement[3]

$$\widetilde{Q}_a(x)^T + \widetilde{Q}_a(x) \leq 0. \tag{21}$$

---
[2] See [5] for a detailed proof of this statement.
[3] If these two conditions are matched the overall system, for which the dynamics can be written as $\widetilde{Q}^{-1}(x)\nabla P(x) = -\widetilde{Q}_a^{-1}(x)\nabla P_a(x) = (di_\rho/dt, dv_\sigma/dt)^T$, is then asymptotically stable.

Define

$$\lambda = -1,$$
$$\Pi(x) = diag[2R_\rho^{-1}, 2\widetilde{L}(v_\sigma)R_\rho^{-1}C^{-1}(v_\sigma)].$$

Considering a mixed potential function $P(x)$ fitting the Assumption A4 and reminding that Assumption A1 $\Rightarrow \Lambda = I$, we obtain

$$\widetilde{Q}_a(x) = \begin{bmatrix} -\widetilde{L}(v_\sigma) & -\widetilde{M}(x) + 2L(v_\sigma)R_\rho^{-1} \\ -2R_\rho^{-1}\widetilde{L}(v_\sigma) & -[I-\beta(x)]C(v_\sigma) - 2R_\rho^{-1}\widetilde{M}(x) \end{bmatrix}$$

that, under Assumptions A3 and A5, satisfies (21). We refer to the appendix for a detailed development of the former statement. Furthermore, the storage function candidate becomes

$$\widetilde{P}_a(x) = \frac{1}{2}(R_\rho i_\rho + v_\sigma)^T R_\rho^{-1}(R_\rho i_\rho + v_\sigma) + \\ +\frac{1}{2}v_\sigma{}^T R_\rho^{-1}v_\sigma + i_\rho{}^T\widetilde{L}(v_\sigma)R_\rho^{-1}C^{-1}(v_\sigma)i_\rho \tag{22}$$

that, under Assumption A6, is clearly positive definite.

Remark 3. Assumption A5 is an important condition that can be satisfied for *small* values of the matrix R_ρ^{-1}—which represent the LTI resistors placed in series to each inductor— and/or with a *weak* mutual-coupling action provided by the presence of the matrix $\widetilde{M}(x)$. Since $\widetilde{M}(x)$ depends linearly on the current vector i_ρ—see $\widetilde{M}(x)$ definition provided in Theorem 1—, we can state that for *slow motion* or *well-damped* dynamics, A5 holds.

4 The Inverted Pendulum on a Cart

An interesting example of mechanical system to study is the inverted pendulum with rigid massless rod (of length l) placed on a cart as shown if Fig. 1. It is often used to test the performance of controllers that stabilize the pendulum mass m_2 to its natural unstable equilibrium point through a force F acting just on the cart of mass m_1. The equations describing the dynamics of the to masses could be computed considering as state variables the angular position of the row with the vertical axis θ and the cart distance $z - z_0$ to a fixed reference ($z_0 = 0$). The motion dynamic of each mass can be determined via the Euler-Lagrange equations

$$(m_1 + m_2)\ddot{z} + m_2 l\cos\theta\ddot{\theta} - m_2 l\sin\theta\dot{\theta}^2 = F - R_1\dot{z} \\ m_2 l^2\ddot{\theta} + m_2 l\cos\theta\ddot{z} - m_2 gl\sin\theta = -R_2\dot{\theta} \tag{23}$$

where the generalized coordinates related to the position of each mass and its derivative representing the corresponding velocities are respectively

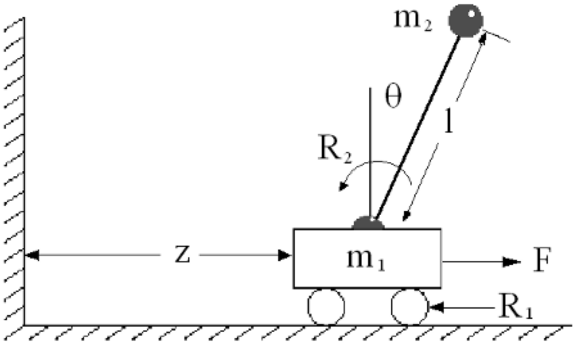

Fig. 1. Inverted pendulum on a cart

$$q = (z, \theta)^T \ , \quad \dot{q} = (\dot{z}, \dot{\theta})^T .$$

Applying the following coordinates transformation[4]

$$\begin{bmatrix} z \\ \theta \end{bmatrix} = \begin{bmatrix} C_1 v_{\sigma 1} \\ f_2^v(v_{\sigma 2}) = \arcsin\left(\frac{v_{\sigma 2}}{K}\right) \end{bmatrix} \ , \quad \begin{bmatrix} \dot{z} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} i_{\rho 1} \\ i_{\rho 2} \end{bmatrix} \tag{24}$$

with $K = -m_2 g l$ and considering that Assumptions A1 and A2 are clearly satisfied we can express the motion equations (23) via the Brayton-Moser framework

$$\widetilde{Q}(x)\dot{x} = \nabla P(x) \tag{25}$$

with

$$\widetilde{Q}(x) = \begin{bmatrix} -(m_1+m_2) & -m_2 l \cos f_2^v(v_{\sigma 2}) & 0 & m_2 l \sin f_2^v(v_{\sigma 2}) i_{\rho_2} C_2(v_{\sigma 2}) \\ -m_2 l \cos f_2^v(v_{\sigma 2}) & -m_2 l^2 & 0 & 0 \\ 0 & 0 & C_1 & 0 \\ 0 & 0 & 0 & C_2(v_{\sigma 2}) \end{bmatrix}$$

and

$$P(x) = -v_s^T i_\rho + \frac{1}{2} {i_\rho}^T R_\rho i_\rho + {i_\rho}^T v_\sigma ,$$

being

$$v_s = (F, 0)^T \ , \quad R_\rho = diag(R_1, R_2) \ , \quad C_2(v_\sigma) = \frac{\partial f_2^v(v_{\sigma 2})}{\partial v_{\sigma 2}}$$

and $C_1 \in \mathbb{R}_+$ an arbitrary constant. Now that we have expressed the mechanical system model by (25) we can use the Theorem 2 in order to get the explicit passivity condition. By choosing

[4] The relation$(\dot{q}, q) \Leftrightarrow (i_\rho, v_\sigma)$ is one-to-one only when θ belongs to the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

$$\lambda = -1 \quad , \quad \Pi(x) = diag[2R_\rho^{-1}, 2\widetilde{L}(v_\sigma)R_\rho^{-1}C^{-1}(v_\sigma)],$$

after some algebraic computations we get the final local condition

$$\left\| \begin{bmatrix} 0 & 2m_2 l \sin f_2^v(v_{\sigma 2}) i_{\rho 2} R_1^{-1} \\ 0 & \frac{[m_2 l \sin f_2^v(v_{\sigma 2})]^2}{m_1 + m_2 - m_2 \cos^2 f_2^v(v_{\sigma 2})} + \frac{i_{\rho 1}}{i_{\rho 2}} m_2 l \sin f_2^v(v_{\sigma 2}) C_2(v_{\sigma 2}) R_2^{-1} \end{bmatrix} \right\| \leq 1 \quad (26)$$

achieved for $C_1 \to \infty$. The former suitable choice of C_1 parameter is arbitrary because it depends on the coordinates transformation we arbitrary fixed. Of course, in order to apply Theorem 2 we have to verify, together with condition (26), that Assumption 6 holds, that means

$$\left\| \begin{bmatrix} 0 & m_2 l \cos f_2^v(v_{\sigma 2}) R_2^{-1} C_2^{-1} \\ 0 & m_2 l^2 R_2^{-1} C_2^{-1}(v_{\sigma 2}) \end{bmatrix} \right\| \geq 0. \qquad (27)$$

From the overlap of (26) and (27), we deduce that

$$\int_0^t F\dot{\hat{i}}_{\rho 1}(\tau) d\tau \geq \widetilde{P}_a(t) - \widetilde{P}_a(0)$$

with $\widetilde{P}_a(x)$ given by (22), holds.

5 Conclusion and Outlooks

Our main purpose in this document was to present an alternative way to describe the dynamics of a large class of (possibly non-)linear mechanical systems within a framework–the Bryton-Moser equations–that relates the power to the trajectories of the system instead of energy, and derive from it sufficient conditions for passivity. This should be consider as a preliminary step towards stabilization of mechanical and electromechanical systems using passivity arguments–as already suggested in [5])–.

Acknowledgements

The work was partially supported through a European Community Marie Curie Fellowship, in the framework of the CTS (contract number: HPMT-CT-2001-00278) and partially financed by the european exchange program Van Gogh.

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Appendix

Here, we show that given Assumptions A3 and A5 of Theorem 2, the positivity of $Q_a(x)$ is established, i.e., (21) holds. Indeed, computing the symmetric part of $Q_a(x)$ we obtain

$$H_a(x) = \begin{bmatrix} \widetilde{L}(v_\sigma) & \widetilde{M}(x) \\ \widetilde{M}^T(x) & [I-\beta(x)]C(v_\sigma) + 2R_\rho^{-1}\widetilde{M}(x) \end{bmatrix}.$$

Then, provided the positivity of $\widetilde{L}(v_\sigma)$ by A3, we compute the Schur's complement of $H_a(x)$ and imposing its positivity we obtain

$$[I-\beta(x)]C(v_\sigma) + 2R_\rho^{-1}\widetilde{M}(x) \geq \widetilde{M}^T(x)\widetilde{L}^{-1}(v_\sigma)\widetilde{M}(x)$$

Let's re-write the above inequality as follows

$$I \geq -2R_\rho^{-1}\widetilde{M}(x)C^{-1}(v_\sigma) + \widetilde{M}^T(x)\widetilde{L}^{-1}(v_\sigma)\widetilde{M}(x)C^{-1}(v_\sigma) + \beta(x),$$

as a consequence of Perron's theorem[5] and reminding that the spectral norm applying on any squared matrix $A \in \mathbb{R}^{r\times r}$ is defined as

$$||A|| = \sqrt{\rho(A^T A)}$$

we have

$$1 \geq \left\| -2R_\rho^{-1}\widetilde{M}(x)C^{-1}(v_\sigma) + \widetilde{M}^T(x)\widetilde{L}^{-1}(v_\sigma)\widetilde{M}(x)C^{-1}(v_\sigma) + \beta(x) \right\|$$

which is true by Assumption A5.

[5] See lemma 8.4.2 of [6].